The anatomy of a Putnam problem

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What is Putnam competition?

- The William Lowell Putnam Mathematical Competition is the premiere competition for undergraduate students in North America.
- More than 500 universities compete in this contest organized by the Mathematical Association of America (MAA).
- The competition takes place in the FIRST Saturday of December.
- The competition was founded in 1927 by Elizabeth Lowell Putnam in memory of her husband William Lowell Putnam, who was an advocate of intercollegiate intellectual competition.
- There are 2 sessions (A1-A6 & B1-B6) of 6 questions each.

Putnam B2-2007.

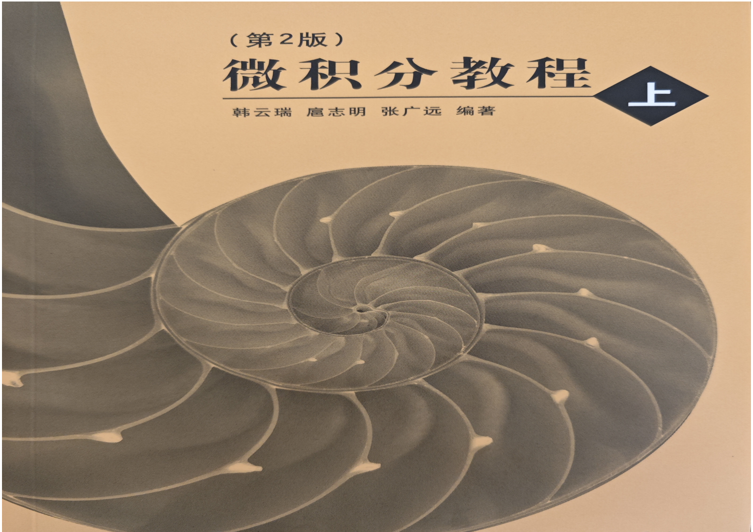
The problem B2-2007 from the prestigious William Lowell Putnam Mathematical Competition reads as following:

Problem (Putnam B2-2007)

Suppose that $f : [0, 1] \to \mathbb{R}$ *has a continuous derivative and that* $\int_0^1 f(x)\,dx = 0$. *Prove that for every* $\alpha \in (0, 1)$,

$$\left| \int_0^{\alpha} f(x)\,dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|$$

Almost identical problems to Putnam B2-2007. The engineering textbook at the Tsinghua University



Almost identical problems to Putnam B2-2007. The engineering textbook at Tsinghua University

5. 设 f 在$[a,b]$上可导，在(a,b)内二阶可导，如果 $f'(a)f'(b)>0$，且 $f(a)=f(b)$，试证 $\exists\,\xi\in(a,b)$，使得 $f''(\xi)=0$.

6. 若 f 在(a,b)可导，则其导函数 $f'(x)$没有第一类间断点.

7. 试举出一个函数 f，它在$(-\infty,+\infty)$上处处可导，其导函数 $f'(x)$在 $x=0$ 处有第二类间断点.

8. 设 $f(x)$在$[0,a]$二阶可导，$|f''(x)|\leqslant M$，$0\leqslant x\leqslant a$. 又设 $f(x)$在$(0,a)$取得极大值. 求证 $|f'(0)|+|f'(a)|\leqslant Ma$.

9. 设 $f(x)$在$[0,1]$处处可导，$f(0)=0$，$f(1)=1$ 且 $f(x)\neq x$. 求证 $\exists\,\xi\in(0,1)$使 $f'(\xi)>1$.

10. 选择 a 与 b，使得 $x-(a+b\cos x)\sin x$ 为 5 阶无穷小$(x\to 0)$，

11. 利用泰勒公式求下列极限：

(1) $\lim\limits_{x\to 0}\dfrac{\sin(\sin x)-\tan(\tan x)}{\sin x-\tan x}$；　　(2) $\lim\limits_{x\to 0^+}\dfrac{e^x-1-x}{\sqrt{1-x}-\cos\sqrt{x}}$；

(3) $\lim\limits_{x\to 0}\dfrac{1}{x^4}\left[\ln(1+\sin^2 x)-6\left(\sqrt[3]{2-\cos x}-1\right)\right]$.

12. 设 $f(x)$在$[a,b]$上二阶可导，证明：$\exists\,x_0\in(a,b)$，使得

$$f(b)-2f\left(\frac{a+b}{2}\right)+f(a)=\frac{(b-a)^2}{4}f''(x_0).$$

13. 设 $f(x)$在区间$[a,b]$上一阶可导，在(a,b)内二阶可导，且 $f'(a)=f'(b)=0$，试证 $\exists\,x_0\in(a,b)$，使得

$$|f''(x_0)|\geqslant\frac{4}{(b-a)^2}|f(b)-f(a)|.$$

14. 设 $f(x)\in C^2[a,b]$，$f(a)=f(b)=0$，试证：

(1) $\max\limits_{a\leqslant x\leqslant b}|f(x)|\leqslant\dfrac{1}{8}(b-a)^2\max\limits_{a\leqslant x\leqslant b}|f''(x)|$；

Almost identical problems to Putnam B2-2007. The 2013 PhD Preliminary examination at the University of Pittsburgh

Ph.D. Preliminary Examination (Analysis)

August, 2013

INSTRUCTIONS: *Do all six problems. In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should contain the necessary details. All problems are worth the same number of points.*

1. Let X be a nonempty set, and for any two functions $f, g \in \mathbb{R}^X := \{h : h \text{ is a function from } X \text{ to } \mathbb{R}\}$, let

$$d(f,g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

Prove that $(\mathbb{R}^X, d)$ is a metric space.

2. Prove that $\sum_{n=1}^{\infty} \frac{nx^n}{x^n+1}$ is not uniformly convergent on $[0,1)$, but it defines a continuous function on $[0,1)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, $a, b \in \mathbb{R}$ and $a < b$. Suppose that $f(a) = f(b) = 0$ and $|f''(x)| \leq 1$ for every $x \in [a,b]$. Prove that

$$|f(x)| \leq \frac{(b-a)^2}{8}$$

for every $x \in [a,b]$.

4. Let $f(x) = f(x_1, x_2, x_3, x_4)$ be a C^1 function from $\mathbb{R}^4$ to $\mathbb{R}$ such that $f(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the origin in $\mathbb{R}^4$. Suppose that

$$(\partial f(\mathbf{0})/\partial x_3, \partial f(\mathbf{0})/\partial x_4) \neq (0,0).$$

Prove that there exist an open neighborhood U of $(0,0)$ in $\mathbb{R}^2$ and two C^1 functions $\phi, \psi : U \to \mathbb{R}$ such that

$$f(s,t,\phi(s,t),\psi(s,t)) = 0 = f(s,t,-\psi(s,t),\phi(s,t))$$

holds for every $(s,t) \in U$.

5. Let $\Omega_1 = [0,2] \times [0,1]$ and $\Omega_2 = [1,3] \times [0,1]$. For each $j \in \{1,2\}$, let $f_j : \Omega_j \to \mathbb{R}$ be Riemann integrable over Ω_j. Define $F : [0,3] \times [0,1] \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} f_1(x,y) & \text{if } (x,y) \in \Omega_1 \backslash \Omega_2; \\ f_2(x,y) & \text{if } (x,y) \in \Omega_2 \backslash \Omega_1; \\ \inf\{f_1(x,y), f_2(x,y)\} & \text{if } (x,y) \in \Omega_1 \cap \Omega_2. \end{cases}$$

Prove that F is Riemann integrable over $[0,3] \times [0,1]$.

6. Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ and $f \in C^\infty(\mathbb{R}^3)$. Suppose that $f(x,y,z) = 0$ for all $(x,y,z) \in \partial\Omega$. Prove that

$$\left| \iiint_\Omega f(x,y,z)\, dV \right| \leq \frac{2\sqrt{5\pi}}{15} \left(\iiint_\Omega |\nabla f(x,y,z)|^2 dV \right)^{1/2}.$$

How to solve a Putnam problem of this caliber?

Following Polya's advice on *How to Solve It*, firstly, we need to understand where is this problem coming from?

- Do you understand all the words used in stating the problem?
- Is there enough information to enable you to find a solution?

Secondly, we need to devise a plan to attack the problem!

- Use direct reasoning!
- Look for a pattern!
- Can we solve a simpler problem? Or a related problem?
- Maybe we have to be ingenious ??!

Thirdly, you can carry out the plan!
Last but not least, one can reflect and look back!

Can we think of a simpler problem?

The following problem is an original question of Polya himself from his famous book with Szego, *Problems in Mathematical Analysis I*:

Problem (Problem 121, pp. 80 (slightly modified))

Assume that the function $f(x)$ is differentiable on $[a, b]$ and that $f(a) = f(b) = 0$. Then we have

$$\int_a^b |f(x)|dx \leq \frac{(b-a)^2}{4} \sup_{a \leq x \leq b} |f'(x)|.$$

The main equation is the following: Do we know a solution (maybe more?!) for this simpler problem?

First solution for Polya's problem.

- The main idea is to apply the mean value theorem on the intervals (a, x) and (x, b), with $x \in (a, b)$. One can write the above integral as the sum of two integrals on the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$.

Mean value theorem applied on the intervals (a, x) and (x, b) give us

$$f(x) = f(a) + f'(c_1)(x - a), c_1 \in (a, x)$$

and

$$f(x) = f(b) + f'(c_2)(x - b), c_2 \in (x, b).$$

This last two equalities will easily imply

First solution for Polya's problem.

$$|f(x)| \leq M(x-a), x \in [a,b]$$

and

$$|f(x)| \leq M(b-x), x \in [a,b]$$

where $M = \sup_{a \leq x \leq b} |f'(x)|$.

Finally, we follow the main idea and integrating on the two subintervals, we obtain

$$\int_a^b |f(x)|dx \leq \int_a^{\frac{a+b}{2}} M(x-a)dx + \int_{\frac{a+b}{2}}^b M(b-x)dx = \frac{(b-a)^2}{4}M.$$

Second "solution" for Polya's problem.

Theorem (Taylor series expansion)

Suppose that f is defined on some open interval I be n times differentiable at the point a and suppose $f^{(n+1)}(x)$ exists on that interval. Then for each $x \neq a$ there is a value c between x and a such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

For example, if $f \in C^2([a,b])$ we have the following Taylor series approximation,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(c), c \in (a,b).$$

Second "solution" for Polya's problem.

Our intention is to apply Taylor's theorem for the antiderivative of f on the intervals $\left[a, \frac{a+b}{2}\right]$ around $x = a$ and $\left[\frac{a+b}{2}, b\right]$ around $x = b$. We have

$$F\left(\frac{a+b}{2}\right) = F(a) + \frac{(b-a)}{2} f(a) + \frac{(b-a)^2}{8} f'(c_1), c_1 \in \left(a, \frac{a+b}{2}\right)$$

and

$$F\left(\frac{a+b}{2}\right) = F(b) + \frac{(a-b)}{2} f(b) + \frac{(a-b)^2}{8} f'(c_2), c_2 \in \left(\frac{a+b}{2}, b\right).$$

By adding the two equalities, we obtain

$$\int_a^b f(x)dx = \frac{(b-a)^2}{8}(f'(c_1) + f'(c_2)) \Longrightarrow$$

Another similar (more difficult?) problem.

$$\left|\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{4} \sup_{a\leq x\leq b} |f'(x)|.$$

We obtained a weaker inequality!!!

Can we think of another similar problem with Putnam B2-2007?

At the Romanian National Mathematical Olympiad in 1984, the following question was posed:

Problem (proposed by Radu Gologan)

Let $f : [0,1] \to \mathbb{R}$ *be differentiable on* $[0,1]$ *such that there exists* $a \in (0,1]$ *with* $\int_0^a f(x)dx = 0$. *Show that*

$$\left|\int_0^1 f(x)dx\right| \leq \frac{1-a}{2} \sup_{x\in(0,1)} |f'(x)|.$$

First solution to the RNMO-1984 problem (mean value theorem).

Can we use similar ideas to Polya's problem to solve the above question?

Indeed, we shall use the idea of applying the mean value theorem again! By a change of variables $x = at$, $t \in (0,1)$, $\int_0^1 f(at)dt = 0$. Again, by the mean value theorem (applied to the function $g(t) = f(tx)$ on $[a,1]$) we have

$$|f(x) - f(ax)| \leq M|x - ax| = M(1-a)x, M = \sup_{x \in (0,1)} |f'(x)|.$$

By applying the triangle inequality combined with the above estimate, we have

$$\left|\int_0^1 f(x)dx\right| \leq \left|\int_0^1 (f(x) - f(ax))dx\right| \leq M(1-a)\int_0^1 xdx = \frac{(1-a)}{2}M.$$

Second solution to the RNMO-1984 problem (Taylor's theorem).

Observe that $\int_0^1 f(x)dx = \int_0^a f(x)dx + \int_a^1 f(x)dx = -\int_a^1 f(x)dx$. Since f is continuous on $[0,1]$, it follows that f has antiderivatives on $[0,1]$, i.e. there exists $F:[0,1]\to\mathbb{R}$ such that $F'(x)=f(x)$. Applying Taylor's formula around the point $x=a$, we derive

$$F(x) = F(a) + \frac{(x-a)}{1!}F'(a) + \frac{(x-a)^2}{2!}F''(c), c \in [0,1].$$

For $x=1$, we have
$F(1) = F(a) + (1-a)f(a) + \frac{(1-a)^2}{2}f'(c), c \in [0,1]$, which gives us

$$\left|\int_1^a f(x)dx\right| = \left|(1-a)f(a) + \frac{(1-a)^2}{2}f'(c)\right|.$$

Second solution to the RNMO-1984 problem (Taylor's theorem).

For $x = 0$, we have

$$F(0) - F(a) = -af(a) + \frac{a^2}{2} f'(c), c \in [0,1],$$

which is equivalent with $-\int_0^a f(x)dx = -af(a) + \frac{a^2}{2} f'(c)$. It follows that $f(a) = \frac{a}{2} f'(c)$, so $|f(a)| \leq \frac{a}{2}|f'(c)| \leq \frac{a}{2} \sup_{x\in(0,1)} |f'(x)|$.

Finally, this implies that

$$\left|\int_0^1 f(x)dx\right| \leq (1-a)\frac{a}{2} \sup_{x\in(0,1)} |f'(x)| + \frac{(1-a)^2}{2} \sup_{x\in(0,1)} |f'(x)| =$$

$$= \frac{(1-a)}{2} \sup_{x\in(0,1)} |f'(x)|.$$

What is convexity?

Maybe we can come up with an ingenious idea! Convexity?!!

Definition

Let $-\infty \leq a < b \leq +\infty$, and let $\varphi : (a, b) \to \mathbb{R}$ be a function. We say that φ is *convex* if

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y), \forall x, y \in (a, b), \lambda \in [0, 1].$$

If the above inequality has reversed sign, then we say that φ is *concave*.

Geometrically, the function

$$\lambda \mapsto ((1-\lambda)x + \lambda y, (1-\lambda)\varphi(x) + \lambda\varphi(y))$$

is a parametrization of a line segment in $\mathbb{R}^2$.

What is convexity?

This line segment has endpoints $(x, \varphi(x))$ and $(y, \varphi(y))$, and is therefore a chord of the graph of φ. Our definitions of concave and convex can be interpreted as follows:

- A function φ is convex if every chord lies above the graph of φ
- A function φ is concave if every chord lies below the graph of φ.

On the other hand, is there a relation between differentiability and convexity? The answer is given by the following

Theorem

Given a twice differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$, we say that φ is convex if $\varphi''(x) \geq 0, \forall x \in \mathbb{R}$. If $\varphi''(x) \leq 0, \forall x \in \mathbb{R}$, then we say that φ is concave.

Third solution to the RNMO-1984 problem (using convexity).

Let M be the maximum of $|f'(x)|$ on $(0,1)$. Let us define the following function

$$g(x) = -\int_0^x f(t)dt + \frac{M}{2}x^2.$$

Clearly, by a short computation of the second derivative, g is convex. This implies

$$g(a) = g((1-a)\cdot 0 + a\cdot 1) \leq (1-a)g(0) + ag(1) = ag(1),$$

which is equivalent with

$$-\int_0^a f(x)dx + \frac{M}{2}a^2 \leq a\left(-\int_0^1 f(x)dx + \frac{M}{2}\right),$$

$$\left|\int_0^1 f(x)dx\right| \leq \frac{M}{2} - \frac{M}{2}a = \frac{(1-a)}{2}M.$$

Official solution for Putnam B2-2007.

Put $B = \max_{0 \le x \le 1} |f'(x)|$ and $g(x) = \int_0^x f(y)dy$. Since $g(0) = g(1) = 0$, the maximum value of $|g(x)|$ must occur at a critical point $y \in (0,1)$ satisfying $g'(y) = f(y) = 0$. We may take $\alpha = y$ hereafter.

Since $\int_0^\alpha f(x)dx = -\int_0^{1-\alpha} f(1-x)dx$, we may assume $\alpha \le \frac{1}{2}$. WLOG, $\int_0^\alpha f(x)dx \ge 0$. From $f'(x) \ge -B$, we have $f(x) \le B(\alpha - x)$, for $0 \le x \le \alpha$, so

$$\int_0^\alpha f(x)dx \le \int_0^\alpha B(\alpha - x)dx = \frac{\alpha^2}{2}B \le \frac{1}{8}B.$$

Second solution (using mean value theorem).

Let $M = \max_{0 \le x \le 1} |f'(x)|$. With the change of variables $x = \alpha t$, we have

$$\left| \int_0^{\alpha} f(x)dx \right| = \left| \alpha \int_0^1 f(\alpha t)dt \right| = \left| \alpha \int_0^1 (f(\alpha t) - f(t))dt \right| \le$$

$$\le \alpha \int_0^1 |f(t) - f(\alpha t)|dt.$$

By using the mean value theorem, one can find $c \in (\alpha t, t)$ such that

$$f(t) - f(\alpha t) = f'(c)(t - \alpha t),$$

hence

Second solution (using mean value theorem).

$$|f(t) - f(\alpha t)| \leq Mt(1-\alpha).$$

We obtain

$$\left|\int_0^\alpha f(x)dx\right| \leq \alpha(1-\alpha)M \int_0^1 tdt \leq \frac{1}{8}M,$$

because $\alpha(1-\alpha) \leq \frac{1}{4}$ for $\alpha \in [0,1]$.

We remark that this solution is very similar to the one given for Problem 2, RNMO-1984!

Third solution (using Taylor series).

Let $F(x) = \int_0^x f(t)dt$. Now, let a be a value in $(0,1)$ at which F^2 is maximized. By the fundamental theorem of calculus we have that F^2 is differentiable and its derivative is equal to $2Ff$. If $F(a) = 0$, then there is nothing to prove. Otherwise $f(a) = 0$. For any x, by Taylor expansion theorem, we have

$$F(x) = F(a) + (x-a)F'(a) + \frac{(x-a)^2}{2}F''(c),$$

with c between a and x.

Third solution (using Taylor series).

For $x = 0$, we have

$$F(0) = F(a) + af(a) + \frac{a^2}{2} f'(c_1),$$

with c_1 between 0 and x.
For $x = 1$, we have

$$F(1) = F(a) + (1-a)f(a) + \frac{(1-a)^2}{2} f'(c_2),$$

with c_2 between a and 1. This imply the following estimates:

$$|F(a)| \leq \frac{1}{2} a^2 f'(c_1)$$

and

$$|F(a)| \leq \frac{1}{2} (1-a)^2 f'(c_2).$$

Third solution (using Taylor series).

Therefore, we have

$$|F(x)| = \sqrt{F^2(x)} \leq \sqrt{F^2(a)} = \sqrt{F(a) \cdot F(a)} \leq$$

$$\leq \sqrt{\frac{1}{2}a^2 f'(c_1)\frac{1}{2}(1-a)^2 f'(c_2)} \leq \frac{1}{2}a(1-a)\max_{0\leq x\leq 1}|f'(x)| \leq$$

$$\leq \frac{1}{8}\max_{0\leq x\leq 1}|f'(x)|,$$

because $\alpha(1-\alpha) \leq \frac{1}{4}$ for any $\alpha \in [0,1]$.

This solution resembles in some sense the "solution" we gave for Polya's problem.

Fourth solution (using convexity!).

Let M be the maximum of $|f'(x)|$ on $[0,1]$. Let us define the following function

$$g(x) = \int_0^x f(t)dt + \frac{M}{2}x^2.$$

A computation of the derivative will give us that g is a convex function. We have

$$g(\alpha) = g((1-\alpha)\cdot 0 + \alpha \cdot 1) \leq (1-\alpha)g(0) + \alpha g(1),$$

which is equivalent to

Fourth solution (using convexity!).

$$\int_0^\alpha f(x)dx + \frac{M}{2}\alpha^2 \leq \alpha \left(\int_0^1 f(t)dt + \frac{M\alpha^2}{2} \right),$$

equivalently

$$\int_0^\alpha f(x)dx \leq \frac{M}{2}(\alpha - \alpha^2) \leq \frac{1}{8}M,$$

where we have used again the fact $\alpha(1-\alpha) \leq \frac{1}{4}$ for any b$\alpha \in [0,1]$.

Fifth solution (convexity again).

One can tweak the previous solution a bit!

Again, let M be the maximum of $|f'(x)|$. Define the following function

$$g(x) = \int_0^x f(t)dt + \frac{M}{2}(x^2 - x).$$

Clearly g is convex and $g(0) = g(1) = 0$. Hence, $g(x) \leq 0$ on the interval $[0, 1]$. This gives us

$$\int_0^x f(t)dt \leq \frac{M}{2}(x - x^2) \leq \frac{M}{8}.$$

Sixth solution (Taylor series again!).

Let $F(\alpha) = \int_0^\alpha f(x)dx$. Taylor expansion theorem for α, evaluated at 1 gives

$$F(1) = F(\alpha) + (1-\alpha)F'(\alpha) + \frac{(1-\alpha)^2}{2}F''(\beta), \beta \in (\alpha, 1)$$

But we know $F(1) = 0$. At the maximum values of $|F(\alpha)|$ we will have $F'(\alpha) = 0$. This gives us $F(\alpha) = -\frac{(1-\alpha)^2}{2}F''(\beta)$

We can assume that the maximum occurs at $\alpha \geq 1/2$ by symmetry, so we have $|F(\alpha)| \leq \frac{1}{8}|F''(\beta)| \leq \frac{1}{8}\max_{0\leq x\leq 1}|F''(x)|$ which is the desired result.

Seventh solution (integration by parts).

Integrating by parts, we get the formula

$$\int_0^{\alpha} f(x)\,dx = \alpha f(0) - \int_0^{\alpha} (x-\alpha) f'(x)\,dx.$$

In particular (when $\alpha = 1$), we have

$$\int_0^{1} f(x)\,dx = f(0) - \int_0^{1} (x-1) f'(x)\,dx.$$

Combining these two equations gives

$$\begin{aligned} -\int_0^{\alpha} f(x)\,dx &= \alpha \int_0^{1} f(x)\,dx - \int_0^{\alpha} f(x)\,dx \\ &= \int_0^{\alpha} (x-\alpha) f'(x)\,dx - \alpha \int_0^{1} (x-1) f'(x)\,dx. \end{aligned}$$

Seventh solution (integration by parts).

$$= \alpha \int_\alpha^1 (1-x) f'(x)\, dx + (1-\alpha) \int_0^\alpha x f'(x)\, dx.$$

Taking the absolute value of both sides gives

$$\left| \int_0^\alpha f(x)\, dx \right| \leq \alpha \int_\alpha^1 (1-x)|f'(x)|\, dx + (1-\alpha) \int_0^\alpha x|f'(x)|\, dx.$$

Let M be the maximum value of $|f'(x)|$. Then we get

$$\begin{aligned} \left| \int_0^\alpha f(x)\, dx \right| &\leq \alpha M \int_\alpha^1 (1-x)\, dx + (1-\alpha) M \int_0^\alpha x\, dx \\ &= \frac{1}{2}\alpha(1-\alpha)^2 M + \frac{1}{2}(1-\alpha)\alpha^2 M \\ &= \frac{1}{2}\alpha(1-\alpha) M \leq \frac{1}{8} M. \end{aligned}$$

Eight solution (integration by parts again!).

Let $F(a) = \int_0^a f(x)dx$ and $M = \max\limits_{0\leq x\leq 1} |f'(x)|$. We must show that $-\frac{M}{8} \leq F(a) \leq \frac{M}{8}$ for all $a \in [0,1]$. Since $F'(a) = f(a)$, we see that we need to check the above inequality as the points where $f(a) = 0$, with $0 < a < 1$.

The main idea is to express $F(a)$ in two ways using integration by parts. Indeed, we have

$$F(a) = \int_0^a f(x)dx = \int_0^a x' f(x)dx = -\int_0^a xf'(x)dx,$$

and

$$F(a) = \int_0^1 f(x)dx - \int_a^1 f(x)dx = -\int_a^1 f(x)dx =$$

Eight solution (integration by parts again!).

$$= \int_a^1 (1-x)' f(x) dx = -\int_a^1 (1-x) f'(x) dx.$$

This implies that

$$|F(a)| \leq \frac{a^2}{2} M, |F(a)| \leq \frac{(1-a)^2}{2} M.$$

Multiplying the above inequalities, and $\alpha(1-\alpha) \leq \frac{1}{4}, \alpha \in [0,1]$, we have
$|F(a)|^2 \leq \dfrac{1}{4 \cdot 16} M^2$, and hence, $|F(a)| \leq \dfrac{1}{8} M$.

Ninth solution.

One final convexity tweak! Let $M = \max_{0 \le x \le 1} |f'(x)|$, and consider the following function,

$$g(x) = \int_0^x f(t)\,dt + \frac{M}{8}(2x-1)^2.$$

A computation of the second derivative shows that g is a convex function. This time we have $g(0) = g(1) = \frac{M}{8}$, so $g(x) \le \frac{M}{8}$.

Tenth solution (Rolle's theorem).

For each fixed $x \in (0,1)$, define $A(x)$ so that

$$\int_0^x f(t)dt = \frac{A(x)}{2}(x - x^2).$$

This is just solving a linear equation; in fact we can write

$$A(x) = \frac{2}{x - x^2}\int_0^x f(t)dt.$$

Now define $g(y) = \int_0^y f(t)dt - \frac{A(x)}{2}(y - y^2)$.
Note that $g'(y) = f(y) - \frac{A(x)}{2} + A(x)y$ and $g''(y) = f'(y) + A(x)$.

Tenth solution (Rolle's theorem).

Clearly, $g(y) = 0$ when $y = 0, y = x$, and $y = 1$. Therefore, by Rolle's theorem, there exist numbers $a \in (0, x)$ and $b \in (x, 1)$ such that $g'(a) = g'(b) = 0$. Then a second application of Rolle's theorem shows that there exists $\xi \in (a, b) \subset (0, 1)$ such that $g''(\xi) = 0$. But that implies that $f'(\xi) = -A(x)$.

So we conclude that there exists $\xi \in (0, 1)$ such that

$$\int_0^x f(t)dt = -\frac{f'(\xi)}{2}(x - x^2).$$

It follows from this that

$$\left|\int_0^x f(t)\,dt\right| \leq \frac{1}{2}(x - x^2)\max|f'(t)| \leq \frac{1}{8}\max|f'(t)|.$$

What is Lagrange interpolation polynomial?

Problem. Given the values of the function $f(x)$ at $n+1$ distinct points $x_0, x_1, \dots, x_n$ and $f_0, f_1, \dots, f_n$ where $f_i = f(x_i)$ for $i = 0, 1, \dots, n$. Find a polynomial of degree n, $P(x)$ such that $P(x_i) = f_i$, $i = 0, 1, \dots, n$.

The simplest case is $n = 2$. That is, given x_0, x_1 and f_0, f_1, we want to find a polynomial of degree 1 such that $P(x_0) = f_0$ and $P(x_1) = f_1$.
The main idea comes from the equation of a line as follows,

$$
\begin{aligned}
y = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) &= f_0 + (f_1 - f_0)\frac{x - x_0}{x_1 - x_0} \\
&= f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}.
\end{aligned}
$$

Lagrange interpolation polynomial.

Denote $L_0(x)$ and $L_1(x)$ be the first degree polynomials,

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

These polynomials satisfy the relations

$$L_0(x_0) = 1, L_0(x_1) = 0,$$
$$L_1(x_0) = 0, L_1(x_1) = 1.$$

So, we can write the polynomial as

$$P(x) = f_0 L_0(x) + f_1 L_1(x).$$

(unique linear polynomial passing through the points (x_0, f_0) and (x_1, f_1)
We have $P(x_0) = f_0 L_0(x_0) + f_1 L_1(x_0) = f_0$ and
$P(x_1) = f_0 L_0(x_1) + f_1 L_1(x_1) = f_1$

Lagrange interpolation polynomial.

The general case for $n+1$ points is as follows. Given $x_0, x_1, \ldots, x_n$ and $f_0, f_1, \ldots, f_n$, first we construct the special polynomials $L_k(x)$ such that $L_k(x_i) = 0$, for $i \neq k$ and $L_k(x_k) = 1$. These polynomials are zero at all points except one.

A polynomial of degree n which is zero at all points except x_k is given by

$$(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n).$$

If we want that polynomial to have the value 1 at x_k we must divide by its values at x_k. In other words, we define the basic Lagrange polynomial of degree n,

$$L_k(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)}.$$

Lagrange interpolation polynomial. Lagrange interpolation theorem.

The polynomial

$$P(x) = f_0 L_0(x) + f_1 L_1(x) + \ldots + f_n L_n(x).$$

is called the *n*th Lagrange interpolating polynomial. Also, $P(x_k) = f_k$ for $k = 0, 1, \ldots, n$.

Theorem

Suppose that $x_0, x_1, \ldots, x_n$ are $n+1$ distinct points in the interval $[a, b]$ and $f(x)$ has $n+1$ continuous derivatives. Then for each $x \in [a, b]$ there exists $c_x \in (a, b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0) \ldots (x - x_n).$$

Eleventh solution (Lagrange interpolation theorem).

Let $F(t) = \int_a^t f(x)dx$, and consider P be the Lagrange interpolation polynomial for F at $x = a$ and $x = b$. By the approximation theorem by Lagrange interpolation, for all $t \in [a, b]$ there exists $c_t \in (a, b)$ such that

$$F(t) - P(t) = \frac{(t-a)(t-b)}{2} F''(c_t).$$

Note that P is the zero polynomial since $P(a) = F(a)$ and $P(b) = F(b) = \int_a^b f(x)dx = 0$. Therefore, we have

$$\left| \int_a^t f(x)dx \right| = |F(t) - P(t)| = \frac{(t-a)(b-t)}{2} |f'(c_t)| \leq \frac{(t-a)(b-t)}{2} M,$$

where $M = \sup_{t \in [a,b]} |f'(t)|$.

Eleventh solution.

Again, we obtained the inequality

$$\left|\int_a^t f(x)dx\right| = \frac{(t-a)(b-t)}{2}|f'(c_t)| \leq \frac{(t-a)(b-t)}{2}M,$$

where $M = \sup_{t\in[a,b]} |f'(t)|$.

By the elementary arithmetic-geometric mean inequality (AM-GM) we have

$$\frac{(t-a)(b-t)}{2} \leq \left(\frac{a+b}{2}\right)^2,$$

and finally, this gives us

$$\left|\int_a^t f(x)dx\right| \leq \left(\frac{a+b}{2}\right)^2 \frac{M}{2},$$

and for $a = 0$ and $b = 1$ our problem folllows immediately.

A variation of Putnam B2-2007

Putnam B2-2007-variation. Suppose that $f \in C^2(\mathbb{R})$ and that $f(1) = \int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)dx \right| \le \frac{2}{81} \max_{0 \le x \le 1} |f''(x)|.$$

Thank you for your attention!!!